Chapter 6

1. (i) We have

$$\begin{aligned} & u_\infty''(x) = 0 \quad \Rightarrow \quad u_\infty(x) = C_1 x + C_2, \\ & u_\infty(0) = 2, \quad u_\infty(1) = -2 \quad \Rightarrow \quad C_2 = 2, \quad C_1 + C_2 = -2 \\ \Rightarrow \quad & C_1 = -4 \quad \Rightarrow \quad u_\infty(x) = -4x + 2. \end{aligned}$$

(ii) As in (i),

$$\begin{aligned} & u_\infty''(x) = 0 \quad \Rightarrow \quad u_\infty(x) = C_1 x + C_2, \\ & u_\infty(0) = -1, \quad u_\infty'(1) = 4 \quad \Rightarrow \quad C_2 = -1, \quad C_1 = 4 \\ \Rightarrow \quad & u_\infty(x) = 4x - 1. \end{aligned}$$

(iii) Similarly,

$$\begin{aligned} & u_\infty''(x) = 0 \quad \Rightarrow \quad u_\infty(x) = C_1 x + C_2, \\ & u_\infty'(0) = 3, \quad u_\infty(1) = 1 \quad \Rightarrow \quad C_1 = 3, \quad C_1 + C_2 = 1 \\ \Rightarrow \quad & C_2 = -2 \quad \Rightarrow \quad u_\infty(x) = 3x - 2. \end{aligned}$$

(iv) As above,

$$\begin{aligned} & u_\infty''(x) = 0 \quad \Rightarrow \quad u_\infty(x) = C_1 x + C_2, \\ & u_\infty'(0) = 0, \quad u_\infty'(1) = 0 \quad \Rightarrow \quad C_1 = 0, \quad \Rightarrow \quad u_\infty(x) = C_2. \end{aligned}$$

By (6.1), we now have

$$\int_0^1 (x+1)\,dx = \int_0^1 C_2\,dx \quad \Rightarrow \quad \big[\tfrac{1}{2}\,x^2 + x\big]_0^1 = \tfrac{3}{2} = C_2 \quad \Rightarrow \quad u_\infty(x) = \tfrac{3}{2}.$$

(v) Once more,

$$\begin{aligned} & u_\infty''(x) = 0 \quad \Rightarrow \quad u_\infty(x) = C_1 x + C_2, \\ & u_\infty(0) = 1, \quad u_\infty'(1) = 3\big[u_\infty(1) - 4\big] \\ \Rightarrow \quad & C_2 = 1, \quad C_1 = 3(C_1 + C_2 - 4) \quad \Rightarrow \quad C_1 = \tfrac{9}{2} \\ \Rightarrow \quad & u_\infty(x) = \tfrac{9}{2}\,x + 1. \end{aligned}$$

2. (i) We have

$$\begin{aligned}
&u''_\infty(x) - 6x + 3 = 0 \quad \Rightarrow \quad u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_1 x + C_2,\\
&u_\infty(0) = 2, \quad u_\infty(1) = -2 \quad \Rightarrow \quad C_2 = 2, \quad 1 - \tfrac{3}{2} + C_1 + C_2 = -2\\
\Rightarrow \quad & C_1 = -\tfrac{7}{2} \quad \Rightarrow \quad u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 - \tfrac{7}{2}\,x + 2.
\end{aligned}$$

(ii) From the given problem we see that

$$\begin{aligned}
&u''_\infty(x) - 6x + 3 = 0 \quad \Rightarrow \quad u'_\infty(x) = 3x^2 - 3x + C_1\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_1 x + C_2,\\
&u_\infty(0) = -1, \quad u'_\infty(1) = 4 \quad \Rightarrow \quad C_2 = -1, \quad C_1 = 4\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + 4x - 1.
\end{aligned}$$

(iii) Once again,

$$\begin{aligned}
&u''_\infty(x) - 6x + 3 = 0 \quad \Rightarrow \quad u'_\infty(x) = 3x^2 - 3x + C_1\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_1 x + C_2,\\
&u'_\infty(0) = 3, \quad u_\infty(1) = 1 \quad \Rightarrow \quad C_1 = 3, \quad 1 - \tfrac{3}{2} + C_1 + C_2 = 1\\
\Rightarrow \quad & C_2 = -\tfrac{3}{2} \quad \Rightarrow \quad u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + 3x - \tfrac{3}{2}.
\end{aligned}$$

(iv) Here

$$\begin{aligned}
&u''_\infty(x) - 6x + 3 = 0 \quad \Rightarrow \quad u'_\infty(x) = 3x^2 - 3x + C_1\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_1 x + C_2,\\
&u'_\infty(0) = 0, \quad u'_\infty(1) = 0 \quad \Rightarrow \quad C_1 = 0\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_2,\\
&\int_0^1 x^2\,dx = \int_0^1 \left(x^3 - \tfrac{3}{2}\,x^2 + C_2\right) dx\\
\Rightarrow \quad & \left[\tfrac{1}{3}\,x^3\right]_0^1 = \left[\tfrac{1}{4}\,x^4 - \tfrac{1}{2}\,x^3 + C_2 x\right]_0^1\\
\Rightarrow \quad & \tfrac{1}{3} = -\tfrac{1}{4} + C_2 \quad \Rightarrow \quad C_2 = \tfrac{7}{12}\\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + \tfrac{7}{12}.
\end{aligned}$$

(v) Finally,

$$\begin{aligned}
& u_\infty''(x) - 6x + 3 = 0 \quad \Rightarrow \quad u_\infty'(x) = 3x^2 - 3x + C_1 \\
\Rightarrow \quad & u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + C_1 x + C_2, \\
& u_\infty(0) = 1, \quad u_\infty'(1) = 3\big[u_\infty(1) - 4\big] \\
\Rightarrow \quad & C_2 = 1, \quad C_1 = 3\big(1 - \tfrac{3}{2} + C_1 + C_2 - 4\big) \\
\Rightarrow \quad & C_1 = \tfrac{21}{4} \quad \Rightarrow \quad u_\infty(x) = x^3 - \tfrac{3}{2}\,x^2 + \tfrac{21}{4}x + 1.
\end{aligned}$$

3. (i) We have

$$\begin{aligned}
& u_\infty''(x) + \gamma x - 1 = 0 \quad \Rightarrow \quad u_\infty'(x) = -\tfrac{1}{2}\,\gamma x^2 + x + C_1 \\
\Rightarrow \quad & u_\infty(x) = -\tfrac{1}{6}\,\gamma x^3 + \tfrac{1}{2}\,x^2 + C_1 x + C_2, \\
& u_\infty'(0) = 0, \quad u_\infty'(1) = 0 \quad \Rightarrow \quad C_1 = 0, \quad -\tfrac{1}{2}\,\gamma + 1 = 0 \\
\Rightarrow \quad & \gamma = 2 \quad \Rightarrow \quad u_\infty(x) = -\tfrac{1}{3}\,x^3 + \tfrac{1}{2}\,x^2 + C_2, \\
& \int_0^1 x^2\,dx = \int_0^1 \big(-\tfrac{1}{3}\,x^3 + \tfrac{1}{2}\,x^2 + C_2\big)\,dx \\
\Rightarrow \quad & \big[\tfrac{1}{3}x^3\big]_0^1 = \big[-\tfrac{1}{12}\,x^4 + \tfrac{1}{6}\,x^3 + C_2 x\big]_0^1 \\
\Rightarrow \quad & \tfrac{1}{3} = \tfrac{1}{12} + C_2 \quad \Rightarrow \quad C_2 = \tfrac{1}{4} \quad \Rightarrow \quad u_\infty(x) = -\tfrac{1}{3}\,x^3 + \tfrac{1}{2}\,x^2 + \tfrac{1}{4}.
\end{aligned}$$

(ii) As above,

$$\begin{aligned}
& u_\infty''(x) + 4x - \gamma = 0 \quad \Rightarrow \quad u_\infty'(x) = -2x^2 + \gamma x + C_1 \\
\Rightarrow \quad & u_\infty(x) = -\tfrac{2}{3}\,x^3 + \tfrac{1}{2}\,\gamma x^2 + C_1 x + C_2, \\
& u_\infty'(0) = 0, \quad u_\infty'(1) = 0 \quad \Rightarrow \quad C_1 = 0, \quad -2 + \gamma = 0 \\
\Rightarrow \quad & \gamma = 2 \quad \Rightarrow \quad u_\infty(x) = -\tfrac{2}{3}\,x^3 + x^2 + C_2, \\
& \int_0^1 x^2\,dx = \int_0^1 \big(-\tfrac{2}{3}\,x^3 + x^2 + C_2\big)\,dx \\
\Rightarrow \quad & \big[\tfrac{1}{3}\,x^3\big]_0^1 = \big[-\tfrac{1}{6}\,x^4 + \tfrac{1}{3}\,x^3 + C_2 x\big]_0^1 \\
\Rightarrow \quad & \tfrac{1}{3} = \tfrac{1}{6} + C_2 \quad \Rightarrow \quad C_2 = \tfrac{1}{6} \quad \Rightarrow \quad u_\infty(x) = -\tfrac{2}{3}\,x^3 + x^2 + \tfrac{1}{6}.
\end{aligned}$$

(iii) In the usual way,

$$
\begin{aligned}
& u_\infty''(x) + \gamma x^2 + x = 0 \quad \Rightarrow \quad u_\infty'(x) = -\tfrac{1}{3}\,\gamma x^3 - \tfrac{1}{2}\,x^2 + C_1 \\
\Rightarrow \quad & u_\infty(x) = -\tfrac{1}{12}\,\gamma x^4 - \tfrac{1}{6}\,x^3 + C_1 x + C_2, \\
& u_\infty'(0) = 0, \quad u_\infty'(1) = 0 \quad \Rightarrow \quad C_1 = 0, \quad -\tfrac{1}{3}\,\gamma - \tfrac{1}{2} = 0 \\
\Rightarrow \quad & \gamma = -\tfrac{3}{2} \quad \Rightarrow \quad u_\infty(x) = \tfrac{1}{8}\,x^4 - \tfrac{1}{6}\,x^3 + C_2, \\
& \int_0^1 x^2\,dx = \int_0^1 \left(\tfrac{1}{8}\,x^4 - \tfrac{1}{6}\,x^3 + C_2\right) dx \\
\Rightarrow \quad & \left[\tfrac{1}{3}\,x^3\right]_0^1 = \left[\tfrac{1}{40}\,x^5 - \tfrac{1}{24}\,x^4 + C_2 x\right]_0^1 \\
\Rightarrow \quad & \tfrac{1}{3} = -\tfrac{1}{60} + C_2 \quad \Rightarrow \quad C_2 = \tfrac{7}{20} \quad \Rightarrow \quad u_\infty(x) = \tfrac{1}{8}\,x^4 - \tfrac{1}{6}\,x^3 + \tfrac{7}{20}.
\end{aligned}
$$